



# Stable matching in a common generalization of the marriage and assignment models

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## Abstract

In the theory of two-sided matching markets there are two well-known models: the marriage model (where no money is involved) and the assignment model (where payments are involved). Roth and Sotomayor, *Two-Sided Matching*, Cambridge University Press, Cambridge, 1990, asked for an explanation for the similarities in behavior between those two models. We address this question by introducing a common generalization that preserves the two important features: the existence of a stable outcome and the lattice property of the set of stable outcomes. © 2000 Elsevier Science B.V. All rights reserved.

## Résumé

Dans la théorie des ‘agences matrimoniales à deux niveaux’ il existe deux modèles bien connus: le modèle des ‘mariages’ (sans transactions financières) et celui des ‘allocations’ (avec paiement). Roth et Sotomayor, *Two-Sided Matching*, Cambridge University Press, Cambridge, 1990, ont soulevé le problème de trouver une bonne explication qui rende compte des similarités dans le comportement de ces deux modèles. Nous nous occupons de cette question en introduisant une généralisation commune qui conserve les deux aspects importants: l’existence d’un résultat stable et la structure de treillis de l’ensemble des résultats stables. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The interest in the theory of matching comes from an unusual blend of disciplines. If economic markets are modeled in the language of matching games, then game-theoretical questions arise that often have combinatorial solutions. Hence, this subject is treated both in the economic, game-theoretic and combinatorial literature.

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By a two-sided matching market we mean a situation where there are two distinct sets of agents,  $P$  and  $Q$ , that are to be matched in pairs consisting of one  $P$ -agent and one  $Q$ -agent. Roth and Sotomayor [7] have written a comprehensive survey of the theory of two-sided matching markets, mainly dealing with two kinds of two-sided matching markets, called the *marriage model* and the *assignment model*. In the marriage model, the agents have absolute preferences on their candidates for marriage, as opposed to the assignment model where there is money involved, and the goal of the agents is to get as big a profit as possible from their match. The possibility of monetary transfers in this latter model makes it ‘flexible’ as compared to the more ‘rigid’ marriage model.

Viewing the matching procedure as a game played by the agents, an outcome of the game will be a matching of certain  $P$ -agents with certain  $Q$ -agents, together with a payoff to each agent in case there is money in the model. Roth and Sotomayor cites empirical evidence for the importance of *stability* of an outcome, i.e. absence of ‘blocking pairs’ — pairs of agents that are not matched but who both have incentive to break up with their current match in order to match up with each other instead.

For both models mentioned above, a stable outcome is known to exist. Gale and Shapley [3] presented an algorithm for finding a stable outcome in the marriage model. Demange et al. [2] solved the same problem for the assignment model. It is also well known that the set of stable outcomes in both cases, under certain conditions, is a complete lattice under preferences of all  $P$ -agents. In the marriage model the condition is that preferences be strict. In the assignment game the condition is that only one assignment is optimal; otherwise, we have the lattice property for the payoffs only, not for the underlying assignments. A consequence of the complete lattice property is that there exists a unique  $P$ -optimal outcome, and this is the solution found by both algorithms mentioned.

In the present paper, we address a challenge raised by Roth and Sotomayor [7] as well as by others: to explain the similarities in behavior between the two models. In a recent article, Roth and Sotomayor [8] themselves elaborate on the background of the question and gives a kind of solution. They present a generalized assignment game that contains both the old models as special cases, and they show that it too has the lattice property for payoffs under the condition that the core equals the ‘core under weak domination’ (or equivalently, that the set of stable payoffs equals the set of ‘strongly stable payoffs’), but they cannot guarantee that the set of solutions to their game is nonempty.

Our approach has both similarities and differences to the approach of Roth and Sotomayor. We introduce another common generalization of the two old models, in which we allow for some agents to behave in a way that we call ‘rigid’. It means that they accept only one particular payoff in a given assignment. One may think of it as there being a rule for payoffs recommended by for example a labor union, which some agents feel obliged to follow while others feel no such obligation. This game is not a special case of Roth–Sotomayor’s, nor does it include their game. It is possible to extend their theory to conclude that the set of strongly stable payoffs is a lattice in our game, but in general the set of stable payoffs will be larger. Kaneko

[5] presented another common generalization of the two kinds of markets. Kaneko's model is more complicated and, as far as we can see, not closely related to ours. He proves nonemptiness of the core but does not consider lattice properties or  $P$ -optimal outcomes.

By presenting an algorithm that produces a stable outcome, we prove that the set of stable outcomes is nonempty for our generalized assignment game. The algorithm combines ideas from the algorithms for the two previous models together with a variation of the technique with augmenting paths, well known in matching theory. The algorithm will work for any integral productivity matrix; we then use non-standard analysis to derive the existence of a stable outcome for any real-valued productivity matrix by approximating with non-standard rational numbers.

Furthermore, we introduce a certain non-degeneracy condition that specializes to strict preferences in the marriage model and to uniqueness of the optimal assignment in the assignment game. Under this condition we can prove that set of stable *outcomes* (i.e. payoff combined with assignment) is a complete lattice, and also that the algorithm produces the  $P$ -optimal outcome.

We are grateful to a referee of an earlier version of this paper for pointing us to the recent article of Roth and Sotomayor, and to another referee for giving the reference to Kaneko.

## 2. The old models and the new generalized model

In order to set the context, we present in turn the marriage model, the assignment game and our new generalized game: the RiFle game.

### 2.1. The marriage model

In the marriage model, let  $P$  be a set of men and  $Q$  be a set of women. Every man has preferences on the women, and every woman has preferences on the men. The preferences can be represented by a list of names in order of preference, where a list only the potential partners he or she would prefer being matched with to being single. Let  $\mu$  be an assignment of men to women. If  $\mu$  has two matched pairs  $p_1 \overset{\mu}{\leftrightarrow} q_1$  and  $p_2 \overset{\mu}{\leftrightarrow} q_2$  where  $p_1$  prefers  $q_2$  to  $q_1$  and  $q_2$  prefers  $p_1$  to  $p_2$  then the pair  $p_1, q_2$  is a blocking pair. A stable assignment is an assignment without blocking pairs.

Gale and Shapley [3] presented a nice algorithm for finding a stable assignment in this model. Briefly, the algorithm runs over a finite number of time steps; in each step every man proposes to the woman who is most preferred on his list of preferences. Each woman tentatively engages the most preferred of her proposers and reject the rest of them. The rejected men remove the rejecting woman from the top of their list, and in the next time step they proceed to the next one. The engaged man stays, and proposes to the same woman at the next time step too. When every woman has at most one proposer, the stable assignment has been obtained.

### 2.2. The assignment game

In the assignment game of Shapley and Shubik [9], as described in Roth and Sotomayor [7], let  $P = \{p_1, \dots, p_n\}$  be a set of employers and  $Q = \{q_1, \dots, q_n\}$  a set of workers. Every pair  $(p_i, q_j)$  has a potential productivity  $\alpha_{ij} \geq 0$ . If  $\mu$  is an assignment (i.e. a matching of  $P$ -agents to  $Q$ -agents), then its total productivity is

$$T_\mu = \sum_{p_i \overset{\mu}{\leftrightarrow} q_j} \alpha_{ij}.$$

The total productivity is distributed as *payoffs*: every employer  $p_i$  gets payoff  $u_i \geq 0$  and every worker  $q_j$  gets payoff  $v_j \geq 0$ , such that the sum of the payoffs is  $T_\mu$ . An outcome in the assignment game is an assignment  $\mu$  together with compatible payoff vectors  $\mathbf{u}$  and  $\mathbf{v}$ .

An outcome is stable if there are no blocking pairs. This is equivalent to the condition  $u_i + v_j \geq \alpha_{ij}$  for all  $i, j$ , since  $u_i + v_j < \alpha_{ij}$  would mean that the pair  $(p_i, q_j)$  could be better off by leaving their current matches and join forces to obtain the productivity  $\alpha_{ij}$ . Shapley and Shubik proved that the underlying assignment of a stable outcome is always optimal, i.e. it yields maximal total productivity.

The existence of a stable outcome in the assignment game was nonconstructively proved by Shapley and Shubik [9]. Demange et al. [2] gave an algorithm that constructs an outcome. The algorithm is a sort of auction procedure, where the employers raise their bids on the workers until only one bidder is left. We will use ideas also from this algorithm when devising the procedure for a more general model.

It is a natural idea to include *reservation prices* in the model. This means that an agent  $p_i$  or  $q_j$  can choose to stay unmatched, in which case he receives his reservation price, denoted by  $u_i^r \geq 0$  and  $v_j^r \geq 0$ , respectively.

### 2.3. New model: the RiFle assignment game

Let us now define our new model, the *RiFle assignment game* (RI for rigid and FLE for flexible) and see how it includes both the previous ones. Let  $P = \{p_1, \dots, p_n\}$  and  $Q = \{q_1, \dots, q_n\}$  be two equally large sets of agents. (By introduction of dummy agents, this condition can always be met.) To each pair  $(p_i, q_j)$  is assigned a pair of nonnegative real numbers  $(\beta_{ij}, \gamma_{ij})$ .

The interpretation of these numbers is that for each possible partnership  $(p_i, q_j)$ , the potential productivity is  $\alpha_{ij} = \beta_{ij} + \gamma_{ij}$ , and it is somehow prescribed that this productivity should be distributed so that  $p_i$  gets the payoff  $\beta_{ij}$  and  $q_j$  gets the payoff  $\gamma_{ij}$ . An agent who will always demand that this distribution rule be followed strictly (that is, who will accept the prescribed payoff only; no more, no less) is called *rigid*. An agent who is willing to negotiate and use side payments is called *flexible*. In our model, every agent is either rigid or flexible.

In this model it is easy to represent reservation prices  $u_i^r$  and  $v_j^r$ : for every agent we insert a rigid dummy agent whose value to him is the reservation price, but whose value to anyone else is zero.

Let  $\mathcal{R}$  be the set of all *rigid pairs*: pairs  $(p_i, q_j)$  such that at least one of the agents  $p_i$  and  $q_j$  is rigid. Let  $\mathcal{F}$  be the set of all *flexible pairs*: pairs  $(p_i, q_j)$  such that both agents are flexible. Hence,  $\mathcal{R} \cup \mathcal{F} = P \times Q$ .

#### 2.4. A first example

Consider the following productivity matrix where the entries are  $(\beta_{ij}, \gamma_{ij})$ :

	$q_1$	$q_2(r)$
$p_1$	(3, 3)	(3, 6)
$p_2(r)$	(2, 5)	(10, 5)

Here  $p_2$  and  $q_2$  are rigid agents, while  $p_1$  and  $q_1$  are flexible. The only flexible pair is then  $(p_1, q_1)$ . What is a reasonable payoff in this game? Under the matching  $p_1 \overset{\mu}{\leftrightarrow} q_1$ ,  $p_2 \overset{\mu}{\leftrightarrow} q_2$ , the rigid pair will receive  $u_2 = 10$  and 5 while the flexible pair will split the productivity  $3 + 3 = 6$ , for example as  $u_1 = 2$  and  $v_1 = 4$ . However, such an outcome of the game will not be stable: the unmatched pair  $(p_1, q_2)$  is then a blocking pair since by cooperating they would improve their payoffs from 2 to 3 and from 5 to 6, respectively.

#### 2.5. Stability in the RiFle game

We shall now formalize the notion of stable outcomes discussed in the previous example.

**Definition 2.1.** An *outcome* of the RiFle assignment game is a matching  $\mu$  of  $P$ -agents to  $Q$ -agents, together with a *payoff*, i.e. a pair of vectors  $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^n \times \mathbb{R}^n$ . An outcome is denoted by  $(\mathbf{u}, \mathbf{v}; \mu)$ . The outcome is *feasible* if it satisfies:

1. *Individual rationality.* All payoffs must be nonnegative, i.e.  $u_i \geq 0, v_j \geq 0$  for all  $p_i, q_j$ .
2. *Rigidity.* A rigid agent, say  $p_i$ , who is matched, say to  $q_j$ , must get payoff  $u_i = \beta_{ij}$ . If the match  $q_j$  is a flexible agent, then his payoff must be  $v_j \geq \gamma_{ij}$ .
3. *Non-wastefulness.* The sum of all payoffs equals the total productivity, that is,

$$\sum_{p_i \in P} u_i + \sum_{q_j \in Q} v_j = \sum_{p_i \overset{\mu}{\leftrightarrow} q_j} \alpha_{ij}.$$

A *stable outcome* is a feasible outcome satisfying the stability condition of absence of blocking pairs. This can be expressed as

$$u_i + v_j \geq \alpha_{ij} \quad \text{for } (p_i, q_j) \in \mathcal{F},$$

$$u_i \geq \beta_{ij} \text{ or } v_j \geq \gamma_{ij} \quad \text{for } (p_i, q_j) \in \mathcal{R}.$$

A payoff  $(\mathbf{u}, \mathbf{v})$  is *feasible* (resp. *stable*) if  $(\mathbf{u}, \mathbf{v}; \mu)$  is a feasible (resp. *stable*) outcome for some matching  $\mu$ . Such a matching  $\mu$  is said to be *compatible* with the payoff  $(\mathbf{u}, \mathbf{v})$ .

### 2.5.1. Justification for the RiFle assignment game

There are both mathematical and empirical reasons for introducing this new model. To start with the latter, it is a fact that most markets are not entirely free but are regulated in various ways, for example by salary policies, and that certain people are more prone to ignore regulations than other people. The RiFle assignment game is a crude attempt to take this into account in the model.

Now for the mathematical justification. If all agents are flexible, then we are back in the assignment game. On the other hand, if all agents are rigid, then the fixed payoffs are directly translatable to preferences and we are back in the marriage model. It is therefore justified to say that the RiFle assignment game is a common generalization of these models. We will now proceed with showing that the fact that the two old models have certain important properties in common can be explained by showing that these properties are owned also by the RiFle assignment game.

## 3. Summary: results for the RiFle assignment game

It is time to give a brief description of the results we will prove for the RiFle assignment game. Concepts such as core and strong stability that are mentioned below will be defined in Section 4 as we go along.

1. There are no side payments in stable outcomes.
2. The set of stable outcomes equals the core of the game.
3. The set of strongly stable payoffs is a lattice (in general not complete) under  $P$ -preferences.
4. The set of stable outcomes is nonempty, and we present an algorithm that finds a stable outcome for integral matrices.

The algorithm is a heavy piece of work and we treat it, and how to go in the limit to prove the existence of stable outcomes also for real-valued matrices, separately in Section 5. This is as far as we can get without putting some additional restrictions on the model. In Section 6 we will then introduce a non-degeneracy condition on the game, specializing to strict preferences in the marriage model and to uniqueness of the optimal assignment in the assignment game. Given non-degeneracy, we can add the following results to our list.

5. Among stable outcomes, the underlying matching is uniquely determined by the payoff.
6. The set of stable outcomes is a lattice
7. The lattice is complete and hence it has unique  $P$ - and  $Q$ -optimal elements.
8. The algorithm produces the  $P$ -optimal outcome (for integral productivity matrices.)

### 3.1. A comparison with Roth and Sotomayor's approach

Actually, the generalized assignment game of Roth and Sotomayor (henceforth referred to as the *R–S game*) works with utility functions in the vein of Demange and Gale [1]. For simplicity though, we will assume that utility is linearly and uniformly correlated to money, so that we can describe their approach in terms of plain money only. We also change their notations somewhat to facilitate comparison to the RiFle game.

In this version of the R–S game, we have a matrix of pairs of numbers  $(\beta_{ij}, \gamma_{ij})$  and reservation prices  $u_i^r, v_j^r$ . Payments are allowed within matched pairs only, so a feasible outcome  $(\mathbf{u}, \mathbf{v}; \mu)$  must to begin with satisfy  $u_i + v_j = \beta_{ij} + \gamma_{ij}$  if  $p_i \xleftrightarrow{\mu} q_j$  (as well as individual rationality, of course). Given such an outcome, we can define the monetary transfer to  $p_i$  from his match  $q_j$  by  $s_i = u_i - \beta_{ij}$ . In the R–S game it is prescribed that the monetary transfers must belong to some set  $S$ . If  $S = \{0\}$ , then the R–S game specializes to the totally rigid RiFle game, i.e. the marriage model. If  $S = \mathbb{R}$  then it specializes to the totally flexible RiFle game, i.e. the assignment game. (As we will see below, there will be no side payments in the RiFle game either, although they are not a priori ruled out).

Roth and Sotomayor assume that all monetary transfers belong to the same set  $S$ , as opposed to the situation in the RiFle game where the monetary transfer in rigid pairs must be zero, while in flexible pairs it can be any real number. However, we have found that the R–S game can be generalized in the same direction, with a separate set  $S_{ij}$  of possible monetary transfers for every possible pair in a matching, without affecting the proofs of Roth and Sotomayor!

The main results proved for the R–S game is that the set  $\mathcal{S}$  of stable payoffs is a compact lattice under the following three assumptions:

1.  $\mathcal{S}$  is non-empty,
2.  $\mathcal{S}$  is closed, and
3.  $\mathcal{S}$  coincides with  $\mathcal{S}^s$ , the set of strongly stable payoffs.

As we will discuss more in detail later, an outcome is strongly stable if it has no weak blocking pair, i.e. a pair such that one agent prefers it to his current match while the other agent is indifferent.

Assumption 3 is the crucial one, and it does not usually hold for the RiFle game. However, what Roth and Sotomayor really prove, although they never explicitly state it, is that *the set  $\mathcal{S}^s$  of strongly stable payoffs in the R–S game is always a lattice* (possibly empty or not closed). Hence, this is true for the RiFle game too, as we discuss in Section 4.3.

## 4. Results without non-degeneracy conditions

We will present our results in the order of the list in the summary, starting with the two easy observations.

#### 4.1. Side payments

Though flexible agents are not a priori unwilling to engage in the practice of side payments, there will be no such thing in a stable outcome.

**Proposition 4.1.** *If  $p_i$  is matched to  $q_j$  in a stable outcome, then their joint payoff equals their joint productivity, i.e.  $u_i + v_j = \alpha_{ij}$ .*

**Proof.** If  $p_i$  is matched to  $q_j$  then necessarily  $u_i + v_j \geq \alpha_{ij}$ , thanks to the rigidity condition for rigid pairs and the stability condition for flexible pairs. Then non-wastefulness forces all these inequalities to be equalities.  $\square$

**Remark.** If the rigidity condition were relaxed so that a flexible agent in a rigid pair could accept also a payment less than his prescribed share, side payments could occur in a stable outcome. For example, consider the following market:

	$q_1(r)$	$q_2(r)$
$p_1$	(3, 3)	(4, 6)
$p_2$	(1, 1)	(10, 5)

Here  $q_1$  and  $q_2$  are rigid agents, while  $p_1$  and  $p_2$  are flexible. If the rigidity condition is relaxed for flexible agents, then

$$p_1 \overset{\mu}{\leftarrow} q_1, \quad p_2 \overset{\mu}{\leftarrow} q_2, \quad u_1 = 5, \quad u_2 = 8, \quad v_1 = 3, \quad v_2 = 5$$

is a stable outcome where  $p_2$  has paid  $p_1$  two units to make him match with  $q_1$  so that  $p_2$  can match with the for him much more profitable agent  $q_2$ .

The rigidity condition we have chosen does not take into account such long-term planning by agents, but rather assumes that the agent  $p_2$  in the above example would prefer getting his rigid share of ten units than his present lot of eight, although this solution would be unstable and in fact he would end up with just one single unit.

Though the lattice analysis depends on our choice of rigidity, the algorithm we will present below is insensitive to this choice; regardless of this definition it will find a stable outcome without side payments!

#### 4.2. The core of the RiFle assignment game

An outcome  $y$  is *dominated* by another outcome  $x$  if there exists a coalition  $A$  of agents such that every member of  $A$  prefers  $x$  to  $y$  and the rules of the game allow them to enforce  $x$  over  $y$ . The *core* of a game is defined as the set of undominated outcomes.

**Proposition 4.2.** *The core of the RiFle assignment game equals the set of stable outcomes.*



**Proof.** It is trivial that outcomes in the core must be individually rational and satisfy rigidity, non-wastefulness and stability. Hence the core is a subset of the stable outcomes. For the other direction, we must show that an outcome outside the core cannot be stable. But a coalition  $C$  cannot enforce obtaining more money than their total productivity under cooperation in pairs within  $C$ , so at least some pair must get at most as much money from the coalition as if they were to split their joint productivity. Since both prefer the coalition, they also prefer the pair. Hence they are a blocking pair.  $\square$

### 4.3. Strong stability

Let us now dig out as much as we can get from the work of Roth and Sotomayor, cf. Section 3.1. First, we give a formal definition of strong stability. Recall that a weak blocking pair is a pair such that one agent prefers it to his current match while the other agent is indifferent. A flexible pair can never be weakly blocking; if one agent is indifferent and the other agent makes a profit, then the latter one can share his profit to make both agents prefer the match. Consequently, it is sufficient to consider rigid pairs.

**Definition 4.3.** An outcome is *strongly stable* if it is stable and contains no weak blocking pair, i.e.

$$u_i = \beta_{ij} \Rightarrow v_j \geq \gamma_{ij} \quad \text{and} \quad v_j = \gamma_{ij} \Rightarrow u_i \geq \beta_{ij} \quad \text{for } (p_i, q_j) \in \mathcal{R}.$$

Similarly we define a *strongly stable payoff*.

Roth and Sotomayor like to assume that the set of strongly stable payoffs coincides with the set of stable payoffs, as they do in the assignment game, as well as in the marriage model when preferences are strict. However, there seems to be no reasonable condition to guarantee the validity of such an assumption when there are both rigid and flexible agents simultaneously. For example, consider the following market, where we give just  $\alpha_{ij}$  for flexible pairs but  $(\beta_{ij}, \gamma_{ij})$  for rigid pairs:

	$q_1$	$q_2(r)$
$p_1$	18	(10, 7)
$p_2$	21	(14, 5)

Here the agent  $q_2$  is rigid while all the others are flexible. This game is non-degenerate in the sense that we will define in Section 6, but nonetheless it has a stable solution that is not strongly stable:

$$p_1 \overset{\mu}{\leftarrow} q_1, \quad p_2 \overset{\mu}{\leftarrow} q_2, \quad u_1 = 10, \quad u_2 = 14, \quad v_1 = 8, \quad v_2 = 5$$

is stable but it is not strongly stable since it has a weak blocking pair,  $(p_1, q_2)$ , where the flexible agent  $p_1$  is indifferent while the rigid agent  $q_2$  would strictly prefer matching with  $p_1$  to his current match with  $p_2$ .

We define the partial order of  $P$ -preferences by  $\mu \geq_P \mu'$  if  $u \geq u'$  and  $v \leq v'$  (componentwise comparisons). Generalizing the R–S game to accept different sets  $S_{ij}$  of monetary transfers for different pairs, the analysis of Roth and Sotomayor implies:

**Proposition 4.4.** *The set of strongly stable payoffs in the RiFle assignment game is a lattice under  $\geq_P$ .*

However, this lattice needs not be compact so we cannot say that it will have a  $P$ -optimal member (in fact, it usually will not).

## 5. There exists a stable outcome of the RiFle assignment game

Our first main result is that the RiFle assignment game always has a stable outcome.

**Theorem 5.1.** *The set of stable outcomes in the RiFle assignment game is non-empty.*

We will prove the theorem by first using non-standard analysis to verify that it is enough to show existence for stable outcomes for integral matrices, and then giving an algorithm that finds a stable outcome for such matrices. In Section 6 we will discuss an additional nice property of the algorithm: if a certain non-degeneracy condition is satisfied, then the outcome found by the algorithm is optimal among stable outcomes for all  $P$ -agents.

### 5.1. Discretization

Our algorithm assumes a discrete version, where all  $\beta_{ij}$ ,  $\gamma_{ij}$ ,  $u_i$  and  $v_i$  are integers. Hence it works also when all numbers are multiples of  $1/N$  for some  $N > 0$ . To prove the theorem for reals, one can approximate the numbers by multiples of  $1/N$  and let  $N$  tend to infinity. The payoff vectors will then converge to some limit vectors that will satisfy the compatibility and stability conditions. To prove this last part has been considered difficult for such algorithms; cf. the book of Hall [4] for one proof for an algorithm solving the assignment problem. However, using the techniques from *non-standard analysis* (that really deserves to be better known) we shall easily prove our case. Any non-standard proof like this one can be transformed to a standard proof, but then it will not look as pretty.

#### 5.1.1. Basic non-standard analysis

For the logical underpinnings of non-standard analysis, we refer to the book by Robinson [6]. In non-standard analysis, we work in a higher-order non-standard model of analysis, denoted by  ${}^*R$ . This is an enlargement of the system  $R$  of real numbers. In  ${}^*R$  we have access to a larger set of numbers than just the reals: we have also *infinite numbers* and *infinitesimal numbers*. Infinite numbers behave as one would expect; in

particular, the sum of two positive infinite numbers is a new positive infinite number. The infinitesimals are infinitely small, i.e. smaller than all positive reals. In this enlarged universe of non-standard numbers, every number  $a$  has a *standard part*, denoted by  $\text{st}(a)$ , which is the unique standard number that differs from  $a$  only by an infinitesimal. We write  $a \approx b$  if the difference between two non-standard numbers  $a$  and  $b$  is infinitesimal. If  $a$  and  $b$  are finite, then  $a \approx b$  implies  $\text{st}(a) = \text{st}(b)$ . The standardization operator preserves equalities as well as non-strict (but not strict) inequalities, i.e.

$$a = b \Rightarrow \text{st}(a) = \text{st}(b), \quad a \geq b \Rightarrow \text{st}(a) \geq \text{st}(b).$$

A main principle of non-standard analysis is that mathematical statements that can be expressed in standard formalism have the same truth value in both the standard model and the enlarged non-standard model. For our particular case, we will consider the statement that for any matrix  $(\beta_{ij}, \gamma_{ij})_{ij}$  of natural numbers there is a stable outcome  $(\mathbf{u}, \mathbf{v}; \mu)$ . In the non-standard model, this is a statement for non-standard natural numbers (which may be infinite) and the numbers in the outcome may also be non-standard. We shall prove that this implies the same statement for real-valued matrices by transforming them to matrices of non-standard natural numbers and then show that the standard part of the outcome is also a stable outcome. We will need the following lemma.

**Lemma 5.2.** *Fix an infinite non-standard natural number  $N$ . Then for any real number  $a$  there exists a non-standard integer  $\hat{a}$  such that  $a \approx \hat{a}/N$ .*

**Proof.** In the standard model we have the following statement: Given a natural number  $N$  and a real number  $a$ , there exists an integer  $\hat{a}$  such that  $|a - \hat{a}/N| < 1/N$ . This statement then must hold in the non-standard model too. In particular, if  $N$  is an infinite natural number then  $1/N$  is infinitesimal and hence  $a \approx \hat{a}/N$ .  $\square$

**Lemma 5.3.** *If the RiFle game has a stable outcome for any productivity matrix of natural numbers, then it does for any productivity matrix of non-negative reals.*

**Proof.** Let  $(\beta_{ij}, \gamma_{ij})_{ij}$  be a productivity matrix of non-negative reals. Fix an infinite non-standard natural number  $N$  and choose non-standard natural numbers  $(\hat{\beta}_{ij}, \hat{\gamma}_{ij})_{ij}$  according to the above lemma. By the hypothesis, there is then a non-standard stable outcome  $(\hat{\mathbf{u}}, \hat{\mathbf{v}}; \mu)$  for the RiFle game defined by  $(\hat{\beta}_{ij}, \hat{\gamma}_{ij})_{ij}$ . Hence, we have non-wastefulness,

$$\sum_{p_i \in P} \hat{u}_i + \sum_{q_j \in Q} \hat{v}_j = \sum_{p_i \xleftrightarrow{\mu} q_j} \hat{\beta}_{ij} + \hat{\gamma}_{ij}, \quad (1)$$

as well as the equalities that define rigidity and the non-strict inequalities that define individual rationality and stability. Divide both sides of all these equalities and inequalities by the infinite non-standard natural number  $N$ . Eq. (1) becomes

$$\sum_{p_i \in P} \hat{u}_i/N + \sum_{q_j \in Q} \hat{v}_j/N = \sum_{p_i \xleftrightarrow{\mu} q_j} \hat{\beta}_{ij}/N + \hat{\gamma}_{ij}/N \approx \sum_{p_i \xleftrightarrow{\mu} q_j} \beta_{ij} + \gamma_{ij}$$

and since the expression to the right is finite, all numbers in the expression to the left must be finite too. Hence, their standard parts exist so we can define  $u_i = \text{st}(\hat{u}_i/N)$  and  $v_i = \text{st}(\hat{v}_i/N)$ . By definition, we have  $\beta_{ij} = \text{st}(\hat{\beta}_{ij}/N)$  and  $\gamma_{ij} = \text{st}(\hat{\gamma}_{ij}/N)$ . Since the standardization operator preserves equalities and non-strict inequalities, we have proved that the outcome  $(\mathbf{u}, \mathbf{v}; \mu)$  is a stable outcome of the real-valued RiFle game.  $\square$

### 5.2. Overview of the algorithm

Given an integral productivity matrix, we will describe an algorithm that finds a stable outcome via a kind of auction mechanism. We begin with the price vector  $\mathbf{v} = [0, \dots, 0]$ . In the course of the algorithm, the prices on the  $Q$ -agents will only increase, never decrease. The algorithm will in each step modify the price vector  $\mathbf{v}$  as well as a map  $\mu : P \rightarrow Q$ , until  $\mu$  is injective, that is, a matching. If, at a given point in the course of the algorithm, we have  $\mu(p_i) = q_j$ , we say that  $p_i$  *proposes* to  $q_j$  at this point. If  $p_i$  proposes to  $q_j$  and at least one of the agents  $p_i$  and  $q_j$  are rigid, we say that this is a *rigid proposal*.

First, we define a function  $f_{ij}(x)$  that tells us the value of  $q_j$  to  $p_i$  if the price on  $q_j$  is  $x$ , that is, if  $q_j$  is to get the payoff  $v_j = x$ . The value will tacitly depend also on the current map  $\mu$ :

$$f_{ij}(x) = \begin{cases} \beta_{ij} + \gamma_{ij} - x & \text{if } (p_i, q_j) \in \mathcal{F}, \\ \beta_{ij} & \text{if } (p_i, q_j) \in \mathcal{R} \text{ and either } x < \gamma_{ij} \text{ or } x = \gamma_{ij} \\ & \text{and } \mu(p_i) = q_j, \\ 0 & \text{if } (p_i, q_j) \in \mathcal{R} \text{ and either } x > \gamma_{ij} \text{ or } x = \gamma_{ij} \\ & \text{and } \mu(p_i) \neq q_j. \end{cases}$$

The dependence on the map  $\mu$  will assure that in a rigid pair, when the price on a  $Q$ -agent  $q_j$  is as high as rigidity allows for a  $P$ -agent  $p_i$ , and  $p_i$  is not currently mapped to  $q_j$ , then he will never be so in the future either, because the value is set to zero. In this way, we bar rejected rigid proposals from ever being renewed.

For each  $p_i$ , let  $D_i$  be the set of potential partners of maximal value to  $p_i$ , that is,

$$D_i := \left\{ q_k : f_{ik}(v_k) = \max_j f_{ij}(v_j) \right\}.$$

The goal of the algorithm is to produce an assignment  $\mu$  and payoff vectors  $\mathbf{u}$  and  $\mathbf{v}$  satisfying the following conditions:

- $q_i \in D_i$  for all  $i$ .
- $u_i \geq 0, v_j \geq 0$  for all  $i, j$ .
- If  $\mu(p_i) = q_j$ , then  $u_i + v_j = \beta_{ij} + \gamma_{ij}$ . If in addition  $(p_i, q_j) \in \mathcal{R}$ , then  $u_i = \beta_{ij}$  and  $v_i = \gamma_{ij}$ .

The second and third conditions say that the payoffs are compatible with the assignment. The first condition implies stability; there can be no blocking pair if every  $p_i$  has his best possible partner.

The algorithm will in each step modify the price vector  $nchb$  and the map  $\mu$  so that it always satisfies  $\mu(p_i) \in D_i$ . (Note that the set  $D_i$  is not fixed but depends on the current price vector  $v$ .) As soon as the map  $\mu$  is injective, the algorithm halts.

As in many matching algorithms, we will use a notion of ‘augmenting paths’. Let us define the relation  $q_j \sim q_k$  if there exists a  $p_i$  such that  $\mu(p_i) = q_j$  and  $q_k \in D_i$ , that is, if  $q_j$  has a proposer that equally well could have proposed to  $q_k$ . Further, let us extend this relation by transitivity, so we say that  $q_{k_1}$  is *connected to*  $q_{k_m}$  if there exists a ‘path’  $q_{k_1} \sim q_{k_2} \sim \dots \sim q_{k_m}$ .

The algorithm is structured into subprocesses. Subprocess A is basically the Gale–Shapley algorithm for finding stable matchings in the marriage model. After subprocess A is completed, every  $Q$ -agent will be subject to at most one rigid proposal. This will be used in subprocess B, which finds augmenting paths from  $Q$ -agents that are already in the image of  $\mu$  to  $Q$ -agents that have either no proposals or a rigid proposal. After subprocess B is completed, there are no such augmenting paths, so if  $\mu$  is still not injective, then there must be some  $Q$ -agent that has several proposers but which is not part of an augmenting path. In subprocess C, prices are increased on all such  $Q$ -agents and all  $Q$ -agents connected to them, and everything is repeated anew.

The algorithm has two important properties: first, the prices  $v$  never decrease; and second, if at some point some  $q_j$  gets a proposer, then  $q_j$  will never again be without proposers during the algorithm. The algorithm will halt when every  $q_j$  has a proposer.

### 5.3. The algorithm

We are given the sets  $P$  and  $Q$  of agents, and the pairs of non-negative integers  $(\beta_{ij}, \gamma_{ij})$  for every pair  $(p_i, q_j)$  of  $P \times Q$ . We will also have a price vector  $v$  and a map  $\mu : P \rightarrow Q$  that will both be modified during the algorithm. In the first step we set  $v := [0, \dots, 0]$  and choose  $\mu$  such that  $\mu(p_i) \in D_i$  for every  $p_i$ . This is possible, since it is obvious from the definition that every  $D_i$  is always nonempty.

#### 5.3.1. Subprocess A

Find all  $q_j$  that have a rigid proposal. For every such  $q_j$ , find a proposer  $p_i$  such that  $\gamma_{ij}$  is maximal, that is,

$$\gamma_{ij} = \max\{\gamma_{kj} : \mu(p_k) = q_j \text{ and } (p_k, q_j) \in \mathcal{R}\}.$$

Set  $v_j := \gamma_{ij}$ . We keep the proposer  $p_i$ , and bar from ever proposing to  $q_k$  again any other  $p_k$  that at this point proposes rigidly to  $q_k$ .

Subsequently, for every  $P$ -agent  $p_i$ , we recompute their sets  $D_i$  after the modification of the price vector  $v$ . A new map  $\mu$  is chosen such that  $\mu(p_i) \in D_i$  for every  $p_i$ . Repeat the process, increasing  $v$  in each run, until nothing changes anymore.

When subprocess A halts, every  $Q$ -agent will be subject to a rigid proposal from at most one  $P$ -agent. However, a flexible  $Q$ -agent may still have several flexible proposers.

### 5.3.2. Subprocess B

Recall that  $q_j \sim q_k$  means that  $q_j$  has a proposer that could have proposed equally well to  $q_k$ , and that  $q_{k_1}$  is connected to  $q_{k_m}$  if there exists a path  $q_{k_1} \sim q_{k_2} \sim \dots \sim q_{k_m}$ . Abusing notation, let  $p_{k_l}$  denote a proposer that could have proposed equally well to  $q_{k_{l+1}}$  instead of to  $q_{k_l}$ . In subprocess B we do the following:

1. Suppose we find a path  $q_{k_1} \sim q_{k_2} \sim \dots \sim q_{k_m}$  such that  $q_{k_1}$  has at least one extra proposer  $p_i$  except for  $p_{k_1}$ , while  $q_{k_m}$  has no proposer at all. Then we modify  $\mu$  by setting  $\mu(p_{k_1}) := q_{k_2}, \dots, \mu(p_{k_{m-1}}) := q_{k_m}$ . For all other  $P$ -agents,  $\mu$  maps as before. This augments the image of  $\mu$  by one agent,  $q_{k_m}$ . Now subprocess A is run again.
2. Suppose we find a path  $q_{k_1} \sim q_{k_2} \sim \dots \sim q_{k_m}$  such that  $q_{k_1}$  has at least one extra proposer  $p_i$  except for  $p_{k_1}$ , while  $q_{k_m}$  is subject to a rigid proposal. (We may here have  $m=1$ .) As in the previous case, we modify  $\mu$  by setting  $\mu(p_{k_1}) := q_{k_2}, \dots, \mu(p_{k_{m-1}}) := q_{k_m}$ . Let  $p_s$  be the rigid proposer of  $q_{k_m}$ . Set  $\mu(p_s)$  to be undefined for the moment, and bar  $p_s$  from ever proposing to  $q_{k_m}$  again. For all other  $P$ -agents,  $\mu$  maps as before. This does not change the image of  $\mu$ , but the set of barred rigid proposals has been augmented. Now subprocess A is run again.

As long as any of these alternatives is possible, the process is repeated. (If several possibilities are open, choose one.)

### 5.3.3. Subprocess C

Let  $M$  be the set of all  $Q$ -agents that are (i) connected to some  $Q$ -agent who has more than one proposer, and (ii) not connected to any  $Q$ -agents that have either no proposal or a rigid proposal. Modify the price vector  $v$  by increasing  $v_j$  by one for all  $q_j \in M$ . Now subprocess B is run again. The whole process is repeated until  $\mu$  has become injective.

## 5.4. Correctness of the algorithm

We will now show that subprocess C, and hence the algorithm, will eventually halt. In each step  $\mu(p_i) \in D_i$  holds. Every  $q_j$  that has ever had a proposer will always have some proposer. The price vector  $v$  sometimes increases, but never decreases. In the first step,  $v = [0, \dots, 0]$ . This means that if some  $q_k$  does not have a proposer at some point, then  $v_k$  is still zero, in which case the value  $f_{ik}(v_k)$  of  $q_k$  for  $p_i$  is greater than zero. When  $q_j$  is a  $Q$ -agent who does have a proposer  $p_i$ , and the price is allowed to increase sufficiently, we will arrive at a non-positive value of  $q_j$  for  $p_i$ ,  $f_{ij}(v_j) \leq 0$ . Then  $q_j \notin D_i$ , so eventually we must have  $q_k \in D_i$  for some  $p_i$ . When subprocess C is used, there must eventually exist a path from some  $q_j$  with several proposers to  $q_k$ . When subprocess B is used,  $q_k$  gets a proposer. In this way all  $Q$ -agents must eventually get proposers, so  $\mu$  is a matching.

The price vector  $v$  is determined by the algorithm, and will be non-negative. Suppose that  $\mu(p_i) = q_j$ . Then if  $(p_i, q_j) \in \mathcal{F}$ , we set  $u_i := \beta_{ij} + \gamma_{ij} - v_j$ . On the other hand, if  $(p_i, q_j) \in \mathcal{R}$ , we set  $u_i := \beta_{ij}$ . The only thing that remains to be checked is that all  $u_i$  are non-negative. But since  $Q$ -agents with no proposers always have had zero price,

they have always been of positive value to all  $P$ -agents, and the same thing must of course hold for the matched pairs.

### 5.5. An example of running the algorithm

Finally, let us illustrate the algorithm with an example. Let  $P = \{p_1, p_2, p_3, p_4, p_5\}$  and  $Q = \{q_1, q_2, q_3, q_4, q_5\}$ , with three rigid agents:  $p_1$ ,  $p_2$  and  $q_1$ . The following matrix of recommended productivity distributions is given.

	$q_1(r)$	$q_2$	$q_3$	$q_4$	$q_5$
$p_1(r)$	(7, 6)	(9, 9)	(4, 9)	(6, 5)	(6, 4)
$p_2(r)$	(8, 5)	(9, 9)	(3, 5)	(7, 7)	(2, 5)
$p_3$	(5, 8)	17	13	13	8
$p_4$	(1, 5)	8	10	9	6
$p_5$	(1, 6)	12	8	9	7

From the beginning we have the price vector  $v = [0, 0, 0, 0, 0]$ . Then the value matrix  $(f_{ij}(v_j))$  at this point is

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$
$p_1$	7	<span style="border: 1px solid black;">9</span>	4	6	6
$p_2$	8	<span style="border: 1px solid black;">9</span>	3	7	2
$p_3$	5	<span style="border: 1px solid black;">17</span>	13	13	8
$p_4$	1	8	<span style="border: 1px solid black;">10</span>	9	6
$p_5$	1	<span style="border: 1px solid black;">12</span>	8	9	7

,  $\mu(P) = [q_2, q_2, q_2, q_3, q_2]$ .

Here for each row the maximum values are boxed. Since in this case the maximum value was unique in each row, there is no freedom of choice for the map  $\mu$ . Now run subprocess A. Only  $q_2$  has rigid proposals; both of the same payoff,  $\gamma_{12} = \gamma_{22} = 9$ . Pick one of these proposers, say  $p_1$ , and bar  $p_2$ . Raise the price on  $q_2$  to 9, so now  $v = [0, 9, 0, 0, 0]$ . Recompute the values and the map:

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$
$p_1$	7	<span style="border: 1px solid black;">9</span>	4	6	6
$p_2$	<span style="border: 1px solid black;">8</span>	0	3	7	2
$p_3$	5	8	<span style="border: 1px solid black;">13</span>	<span style="border: 1px solid black;">13</span>	8
$p_4$	1	-1	<span style="border: 1px solid black;">10</span>	9	6
$p_5$	1	3	8	<span style="border: 1px solid black;">9</span>	7

,  $\mu(P) = [q_2, q_1, q_3, q_3, q_4]$ .

For  $\mu(p_3)$  we had a choice between  $q_3$  and  $q_4$ . Now run subprocess A. Only  $q_1$  has a new rigid proposal, raising the price on  $q_1$  to 5, giving  $v = [5, 9, 0, 0, 0]$ . The values and the map do not change, so we can proceed to subprocess B. None of the conditions in subprocess B are satisfied, so we proceed to subprocess C.

In subprocess C we identify the set  $M = \{q_3, q_4\}$  as being of the desired kind:  $q_3$  has two proposers, it is connected to  $q_4$ , and  $q_4$  has a proposer. They are not connected

to anyone else. Raise the price by one on both  $q_3$  and  $q_4$ , to obtain  $v = [5, 9, 1, 1, 0]$ . Subprocess B still does not kick into action, so we return to subprocess C where the price on  $q_3$  and  $q_4$  is raised another unit, yielding  $v = [5, 9, 2, 2, 0]$ .

	$q_1$	$q_2$	$q_3$	$q_4$	$q_5$
$p_1$	7	<span style="border: 1px solid black;">9</span>	4	6	6
$p_2$	<span style="border: 1px solid black;">8</span>	0	3	7	2
$p_3$	5	8	<span style="border: 1px solid black;">11</span>	<span style="border: 1px solid black;">11</span>	8
$p_4$	1	−1	<span style="border: 1px solid black;">8</span>	7	6
$p_5$	1	3	6	<span style="border: 1px solid black;">7</span>	<span style="border: 1px solid black;">7</span>

$\mu(P) = [q_2, q_1, q_3, q_3, q_4].$

Now, we see that the prices on  $q_3$  and  $q_4$  have been raised enough for  $p_5$  to find it worth considering proposing to  $q_5$  instead. In this situation subprocess B identifies the path  $q_3 \sim q_4 \sim q_5$ , where  $q_3$  has two proposers while  $q_5$  has none. This is an augmenting path, so we change the map accordingly to  $\mu(P) = [q_2, q_1, q_4, q_3, q_5]$ . Now the map is injective, so the algorithm halts. From the price vector we get payoffs  $v = [5, 9, 2, 2, 0]$  and  $u = [9, 8, 11, 8, 7]$ . The reader is encouraged to check that this is indeed a stable outcome, since no unmatched pair can do better by cooperating.

6. Introducing a non-degeneracy condition

Neither in the marriage model nor in the assignment game is the set of stable outcomes always a lattice (although in the assignment game the set of stable *payoffs* is always a lattice). In degenerate cases there may be, for example, two different best possible outcomes. To guarantee the lattice property some non-degeneracy conditions are needed. For the marriage model a sufficient condition is strictness of preferences, which in the RiFe model with all agents rigid translates to

$$\beta_{ij} = \beta_{ij'} \Rightarrow j = j', \quad \gamma_{ij} = \gamma_{i'j} \Rightarrow i = i'.$$

For the assignment game, the stable outcomes form a lattice if and only if there is a unique optimal assignment, that is if there is a unique matching  $\mu$  such that the total productivity  $T_\mu$  is optimal. A sufficient condition for uniqueness of the optimal assignment is that for any two matchings  $\mu$  and  $\mu'$ , and a minimal coalition  $C$  of agents such that all  $C$ -agents are matched to each other in both  $\mu$  and  $\mu'$ , we have that  $\mu$  and  $\mu'$  must either coincide on  $C$  or yield different total productivities on  $C$ , i.e.

$$\sum_{\substack{p_i, q_j \in C \\ p_i \leftrightarrow q_j}} \alpha_{ij} = \sum_{\substack{p_i, q_j \in C \\ p_i \overset{\mu'}{\leftrightarrow} q_j}} \alpha_{ij} \Rightarrow \mu = \mu' \quad \text{on } C.$$

The non-degeneracy condition that we have found sufficient for the lattice property in the RiFe game contains the conditions above as obvious special cases.

**Definition 6.1.** Given a coalition  $C$  of agents and an assignment  $\mu$  (i.e. no payoffs), we say that the total payoff to  $C$  under  $\mu$  is *forced* if all matched pairs of one agent in



$C$  and one agent outside  $C$  are rigid, because due to rigidity and the absence of side payments the payoff to  $C$  must then be

$$\sum_{\substack{p_i \in C \\ p_i \xleftrightarrow{\mu} q_j}} \beta_{ij} + \sum_{\substack{q_j \in C \\ p_i \xleftrightarrow{\mu} q_j}} \gamma_{ij}.$$

(Recall that an unmatched agent who gets his reservation price is represented as being matched to a rigid dummy agent.)

We say that the RiFle assignment game is *non-degenerate* if the following holds for any two matchings  $\mu$  and  $\mu'$ : if  $C$  is a minimal coalition such that the payoff is forced under both  $\mu$  and  $\mu'$ , and the forced payoffs are equal, then  $\mu$  and  $\mu'$  coincide on  $C$ .

In the special case when all agents are flexible, this condition is obviously identical to the earlier one. In the special case when all agents are rigid, the minimal coalitions consist of one agent and the condition says that his payoff is never indifferent over choice of match, i.e. his preferences are strict.

**Remark.** Is the non-degeneracy condition computable? Yes, but in exponential time. The condition translates to the following inequalities, which must hold for any  $k > 0$  and any renumbering of the agents:

1.  $\alpha_{11} + \cdots + \alpha_{kk} \neq \alpha_{21} + \cdots + \alpha_{k,k-1} + \alpha_{1k}$ .
2.  $\alpha_{11} + \cdots + \alpha_{kk} + \gamma_{00} \neq \alpha_{10} + \cdots + \alpha_{k,k-1} + \gamma_{k+1,k}$ .
3.  $\alpha_{11} + \cdots + \alpha_{kk} + \beta_{00} \neq \alpha_{01} + \cdots + \alpha_{k-1,k} + \beta_{k,k+1}$ .
4.  $\alpha_{11} + \cdots + \alpha_{kk} \neq \alpha_{12} + \cdots + \alpha_{k-1,k} + \gamma_{01} + \beta_{k,k+1}$ .

In these expressions, the agents  $p_0$  and  $q_0$  are allowed to be identical to  $p_{k+1}$  and  $q_{k+1}$  respectively.

### 6.1. Determination of matching from payoff

Given a stable outcome in a non-degenerate RiFle game, it is possible to determine the underlying matching from the payoff.

**Proposition 6.2.** *In a non-degenerate game, only one matching is compatible with a stable payoff.*

**Proof.** To begin with, we can directly determine the match of every rigid agent, say  $p_i$  with payoff  $u_i$ , since by rigidity  $u_i = \beta_{ij}$  for some  $q_j$ , and by non-degeneracy all  $\beta_{ij}$  for varying  $j$  are different.

Remaining agents are flexible and matched to each other. By non-degeneracy there is a unique matching of the remaining agents that has the remaining payoff as total productivity.  $\square$

Combining this observation with the result that the set of strongly stable payoffs is a lattice (Proposition 4.4), we have that the set of all *strongly stable outcomes* is a

lattice in the non-degenerate case. But we want to say something even stronger: that the set of all *stable outcomes* is a lattice. One possible way of completing the result would be to show that every outcome that is stable but not strongly stable can be approximated arbitrarily close by a strongly stable outcome. However, this turns out to be as complicated as showing the theorem from first principles without refering to the work of Roth and Sotomayor, so we prefer to do it that way instead.

### 6.2. Lattice property of stable outcomes

We are going to prove the lattice property. By abusing notation, let  $\mu$  and  $\mu'$  denote the outcomes with underlying matchings  $\mu$  and  $\mu'$  and payoffs  $(\mathbf{u}, \mathbf{v})$  and  $(\mathbf{u}', \mathbf{v}')$  respectively. We define the partial order of  $P$ -preferences by  $\mu \geq_P \mu'$  if  $\mathbf{u} \geq \mathbf{u}'$  and  $\mathbf{v} \leq \mathbf{v}'$  (componentwise comparisons).

**Theorem 6.3.** *For a non-degenerate RiFle game, the set of all stable outcomes is a lattice under the partial order  $\geq_P$  of  $P$ -preferences.*

Proving the lattice theorem will take some work. We must find a join  $\mu \vee \mu'$  and a meet  $\mu \wedge \mu'$  under  $\geq_P$ . It is obvious what we would like  $\check{\mu} = \mu \vee \mu'$  to be: it should have payoffs  $\check{u}_i = \max(u_i, u'_i)$  and  $\check{v}_i = \min(v_i, v'_i)$  and a compatible matching. But there is not necessarily a compatible matching!

#### 6.2.1. Why non-degeneracy is needed

Consider the following productivity matrix:

	$q_1$	$q_2(r)$
$p_1(r)$	(4, 5)	(2, 3)
$p_2$	11	(6, 7)

There are two stable outcomes: on one hand,  $p_1 \overset{\mu}{\leftarrow} q_1$  and  $p_2 \overset{\mu}{\leftarrow} q_2$  with payoff  $u_1 = 4$ ,  $u_2 = 6$ ,  $v_1 = 5$ ,  $v_2 = 7$  is a stable outcome; on the other hand  $p_1 \overset{\mu'}{\leftarrow} q_2$  and  $p_2 \overset{\mu'}{\leftarrow} q_1$  with payoff  $u'_1 = 2$ ,  $u'_2 = 6$ ,  $v'_1 = 5$ ,  $v'_2 = 3$  is a stable outcome too. But the payoff given by  $\check{u}_i = \max(u_i, u'_i)$  and  $\check{v}_i = \min(v_i, v'_i)$  is not compatible with any matching.

We shall show that under the non-degeneracy conditions, a compatible matching  $\check{\mu}$  can be found.

#### 6.2.2. Finding a compatible matching

Fix two stable outcomes  $\mu$  and  $\mu'$ . We represent any unmatched agent in  $\mu$  or  $\mu'$  as being matched to a rigid dummy agent, such that the real agent’s payoff is his reservation price, should they cooperate. The dummy agent prefers being matched to being unmatched. If an agent was unmatched in both  $\mu$  and  $\mu'$ , he is now matched to the same dummy agent in both cases.

We start by defining the digraph  $G(\mu, \mu')$  on the vertex set  $P \cup Q$  as the union of the matchings  $\mu$  and  $\mu'$ , i.e. with the following edges:

$$\{(p_i, q_j): p_i \xrightarrow{\mu} q_j \text{ or } p_i \xrightarrow{\mu'} q_j\},$$

labeled by the corresponding payoffs,  $(u_i, v_j)$  or  $(u'_i, v'_j)$ . Since  $\mu$  and  $\mu'$  are matchings, this graph has vertices of degrees zero, one and two only. We now give some edges directions as follows. Direct an edge  $(p_i, q_j)$  towards  $q_j$  if  $p_i$  prefers this edge, i.e. if it is the only edge incident to  $p_i$ , or if the payoff to  $p_i$  from this edge is strictly greater than the payoff from the other edge. No edge becomes bidirected, because if the two agents both prefer being matched according to one outcome they would constitute a blocking pair in the other outcome, contradicting stability.

Hence, we now have a graph where edges are either undirected or directed. Since no degrees are greater than two, all connected components are paths or cycles. If a component is a path, its two endpoints are dummy agents and the last edge at both ends is directed from the dummy towards the real agent.

**Lemma 6.4.** *In a non-degenerate game, a path in  $G(\mu, \mu')$  between two rigid agents cannot consist of indifferent agents only (unless it is an undirected two-cycle).*

**Proof.** Let  $C$  be the set of agents in-between the two rigid agents. Then it is a minimal coalition with forced total payment in both  $\mu$  and  $\mu'$ . If all agents in  $C$  were indifferent, then the forced total payoffs would be equal, contradicting non-degeneracy.  $\square$

Recall that non-degeneracy implies strict preferences of all rigid agents.

**Lemma 6.5.** *In a non-degenerate game, an indifferent (and hence flexible) agent in  $G(\mu, \mu')$  cannot be preferred by a flexible agent.*

**Proof.** Without loss of generality, say that  $p_i \xrightarrow{\mu} q_j$  where  $p_i$  is indifferent and preferred by  $q_j$ . Indifference of  $p_i$  means that  $u_i = u'_i$ , while preference of  $q_j$  means that  $v_j > v'_j$ . Hence  $u'_i + v'_j < u_i + v_j = \alpha_{ij}$ , so if  $q_j$  were flexible then  $p_i$  and  $q_j$  would be a blocking pair in  $\mu'$ . Thus  $q_j$  must be rigid.  $\square$

**Lemma 6.6.** *In a non-degenerate game, all directed edges in a component of  $G(\mu, \mu')$  must be directed in the same way along the path or cycle.*

**Proof.** A directed path in a component must end in an indifferent agent. Thanks to the previous lemma, the last edge in a directed path must then be directed from a rigid agent towards a flexible agent. Suppose we have two directed paths with different directions in a component (a path or a cycle). Then we must have a path segment of indifferent agents whose first and last members are preferred by rigid agents, contradicting the result of Lemma 6.4.  $\square$

This gives us, to begin with, the following result that is well known in the two old models.

**Proposition 6.7.** *If the non-degeneracy condition holds, then the set of unmatched agents is the same in all stable outcomes.*

**Proof.** The previous lemma says that all directed edges must have the same direction along a path in  $G(\mu, \mu')$ , but every path component has differently directed edges at the ends. Hence there can be no path components, so there are no agents that are matched in  $\mu$  but unmatched in  $\mu'$  or vice versa.  $\square$

Now we are ready to prove the lattice theorem.

**Proof of Theorem 6.3.** Since all directed edges in a cycle are directed in the same way, either  $\mu$  or  $\mu'$  has all its directed edges from  $P$ - to  $Q$ -agents, while the other matching has all its directed edges from  $Q$ - to  $P$ -agents. Hence, on this component the first matching is compatible with the payoffs  $\check{u}_i = \max(u_i, u'_i)$  and  $\check{v}_j = \min(v_j, v'_j)$  and the other matching is compatible with  $\hat{u}_i = \min(u_i, u'_i)$  and  $\hat{v}_j = \max(v_j, v'_j)$ . By taking the  $P$ -optimal matching on each component we obtain  $\check{\mu}$ . Since all matched pairs and their payoffs come from either  $\mu$  or  $\mu'$  we must have both individual rationality, rigidity and non-wastefulness in  $\check{\mu}$ . It remains to prove stability of  $\check{\mu}$ . But by a standard argument a blocking pair in  $\check{\mu}$  would be a blocking pair also in either  $\mu$  or  $\mu'$  contradicting the assumption of their stability. For  $\hat{\mu}$  the argument is analogous.

To summarize, we have found a stable matching compatible with the payoff  $(\check{u}, \check{v})$  and by Proposition 6.2 there can be only one such matching.  $\square$

### 6.3. Completeness of the lattice

**Lemma 6.8.** *The set of stable payoffs is compact.*

**Proof.** Let  $\mu$  be a matching and let  $\mathcal{S}_\mu$  be the set of stable payoffs compatible with  $\mu$ . Then  $\mathcal{S}_\mu$  is a compact set, since all the defining relations for feasibility and stability are equations and non-strict inequalities. Consequently  $\mathcal{S}$  is closed, since it is a finite union of compact sets.  $\square$

From this result and the lattice result in the non-degenerate case, we immediately have the following.

**Theorem 6.9.** *If the RiFle game is non-degenerate, then the set of stable outcomes is a compact lattice. In particular, it has unique  $P$ - and  $Q$ -optimal outcomes.*

### 6.4. The algorithm revisited

We now know that when the game is non-degenerate, there is a unique  $P$ -optimal outcome. As we mentioned in the introduction, the classic algorithms for finding a stable outcome in the marriage model and in the integral assignment game do in fact find precisely the  $P$ -optimal solution, and we claim that this is the case also for our new algorithm. The proof idea is quite simple, but some details get technical and we skip a few of them.

**Theorem 6.10.** *If the non-degeneracy condition is satisfied for an integral RiFle game, then the outcome found by the algorithm is optimal among stable outcomes for  $P$ -agents.*

**Proof (sketch).**

Let  $(m_1, m_2, \dots, m_n)$  be the outcomes of the  $Q$ -agents in the  $P$ -optimal matching. We enumerate the steps in the algorithm as follows: Every turn of proposals in subprocess A counts as one step. The same goes for every construction of a path in subprocess B and every application of subprocess C.

Let  $[v_1^{(i)}, v_2^{(i)}, \dots, v_n^{(i)}]$  be the  $Q$ -outcomes at step  $i$  in the algorithm. We want to show that  $v_k^{(i)} \leq m_k$  for all  $i, k$ . We notice that all the  $v_k^{(i)}$  are non-decreasing and can only increase in subprocess A or subprocess C. We now study these two cases. We assume that  $v_k^{(i)} \leq m_k$  for all  $i, k$  and show that  $v_k^{(i+1)} \leq m_k$ .

*Subprocess A.* Suppose  $v_a^{(i+1)} \geq m_a$ . The increase of  $v_a$  comes from  $q_a$  accepting a proposer  $p_b$  at step  $i + 1$ . We claim that there is no stable matching  $\mu$  giving  $q_a$  an outcome smaller than  $v_a^{(i+1)}$ . Indeed, suppose  $\mu(q_a) = p_c$  and  $\mu(p_b) = q_d$ . In this matching we must have  $v_d \geq m_d \geq v_d^{(i)}$ . If  $v_d > v_d^{(i)}$  then  $p_b$  must prefer  $q_a$  to  $q_d$ , given the  $Q$ -outcomes in  $\mu$ , since  $q_a$  at least did not prefer  $q_d$  to  $q_a$  at step  $i$ . But then  $(p_b, q_a)$  blocks  $\mu$ .

The case  $v_d = v_d^{(i)}$  is more complicated. It is possible to show, by using the non-degeneracy conditions, that the only possibility is that  $p_b, q_d$  are flexible,  $q_a$  is rigid and  $q_d \sim q_a$  in subprocess B. Then  $q_d$  must be contained in a set  $M$ , such that  $q_a \notin M$ , and such that  $M$  is *overdemanded* (i.e., the set  $A: p_i \in A \Rightarrow D_i \subseteq M$  contains more elements than  $M$ ) at step  $i - 1$ . We can use the non-degeneracy conditions to show that the path containing  $q_d$  and  $q_a$  is the only augmenting path at step  $i$ . This means that every stable matching with  $v_d = v_d^{(i)}$  must match  $p_b$  to  $q_a$ . This shows that there can be no stable matching  $\mu$  such that  $\mu(p_b) = q_d$ .

*Subprocess C.* At step  $i + 1$  the outcomes for the  $Q$ -agents in an overdemanded set  $M$  are raised one step. Suppose  $q_a \in M$  and  $v_a^{(i+1)} > m_a$ . Then there is a set  $N \subseteq M$  such that  $q_a \in N$  and  $v_k^{(i+1)} = m_k + 1$ , i.e.,  $v_k^{(i)} = m_k$  for all  $q_k \in N$ . (The  $m_k$ 's must, of course, all be integers.) The  $P$ -optimal matching  $\mu_P$  has  $v_s \geq v_s^{(i)}$  for all  $s$  such that  $q_s \notin N$ . It is then possible to show that  $N$  must be overdemanded at these outcomes. Since the stable matching  $\mu_P$  cannot contain an overdemanded set this is impossible.  $\square$

## 7. Concluding remarks

We have introduced and analyzed a new game, the RiFle game, that is a common generalization of the marriage model and the assignment game, in order to explain the similarities of the results for these two different games. Roth and Sotomayor addressed the same problem recently, and we have compared our results. Our game is not a special case of theirs, but we have remarked that their analysis can be extended to our game. However, the assumptions differ. In particular, the key assumption of Roth and Sotomayor that the set of stable and strongly stable outcomes are equal does not hold for our game, but nonetheless we can show that the set of stable outcomes of the RiFle game is a lattice under  $P$ -preferences. An additional feature of our game is the existence of a stable outcome, which we guarantee by presenting an algorithm that finds one (indeed, the  $P$ -optimal one in the non-degenerate case).

On the other hand, Roth and Sotomayor work with utility functions and not just with plain money. It is not clear how utility functions could be included in our model; both the algorithm and the non-degeneracy condition seem to depend on linearity between money and utility.

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